Pseudocharacters of Classical Groups

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Abstract

A GL_d-pseudocharacter is a function from a group Γ to a ring k satisfying polynomial relations which make it "look like" the character of a representation. When k is an algebraically closed field, Taylor proved that GL_d-pseudocharacters of Γ are the same as degree-d characters of Γ with values in k, hence are in bijection with equivalence classes of semisimple representations $\Gamma \to GL_d(k)$. Recently, V. Lafforgue generalized this result by showing that, for any connected reductive group H over an algebraically closed field k of characteristic 0 and for any group Γ, there exists an infinite collection of functions and relations which are naturally in bijection with $H^0(k)$-conjugacy classes of semisimple representations $\Gamma \to H(k)$. In this paper, we reformulate Lafforgue's result in terms of a new algebraic object called an FFG-algebra. We then define generating sets and generating relations for these objects and show that, for all H as above, the corresponding FFG-algebra is finitely presented. Hence we can always define H-pseudocharacters consisting of finitely many functions satisfying finitely many relations. Next, we use invariant theory to give explicit finite presentations of the FFG-algebras for (general) orthogonal groups, (general) symplectic groups, and special orthogonal groups. Finally, we use our pseudocharacters to answer questions about conjugacy vs. element-conjugacy of representations, following Larsen.

1 Introduction

Pseudocharacters were originally introduced for GL_2 by Wiles [13] and generalized to GL_n by Taylor [12]. Taylor's result on GL_n-pseudocharacters is as follows. Let Γ be a group and k be a commutative ring with identity. Define a GL_n-pseudocharacter of Γ over k to be a set map $T : \Gamma \to k$ such that

- $T(1) = n$
- For all $\gamma_1, \gamma_2 \in \Gamma$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$
- For all $\gamma_1, \ldots, \gamma_{n+1} \in \Gamma$,
$$\sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) T_\sigma(\gamma_1, \ldots, \gamma_{n+1}) = 0,$$
where S_{n+1} is the symmetric group on $n+1$ letters, $\operatorname{sgn}(\sigma)$ is the permutation sign of σ, and T_σ is defined by
$$T_\sigma(\gamma_1, \ldots, \gamma_{n+1}) = T(\gamma_{i_1^{(1)}} \cdots \gamma_{i_{r_1}^{(1)}}) \cdots T(\gamma_{i_1^{(s)}} \cdots \gamma_{i_{r_s}^{(s)}})$$
when σ has cycle decomposition $(i_1^{(1)} \ldots i_{r_1}^{(1)}) \ldots (i_1^{(s)} \ldots i_{r_s}^{(s)})$.

If T is a GL_n-pseudocharacter, then define the kernel of T by

$$\ker(T) = \{\eta \in \Gamma | T(\gamma\eta) = T(\gamma) \text{ for all } \gamma \in \Gamma\}$$

Then:

Theorem 1.1 ([12, Theorem 1]). *1. Let $\rho : \Gamma \to GL_n(k)$ be a representation. Then $\operatorname{tr}(\rho)$ is a GL_n-pseudocharacter.*

2. Suppose k is a field of characteristic 0, and let $\rho : \gamma \to GL_n(k)$ be a representation. Then $\ker(\operatorname{tr}(\rho)) = \ker(\rho^{ss})$, where ρ^{ss} denotes the semisimplification of ρ.

3. Suppose k is an algebraically closed field of characteristic 0. Let $T : \gamma \to k$ be a GL_n-pseudocharacter. Then there is a semisimple representation $\rho : \Gamma \to GL_n(k)$ such that $\operatorname{tr}(\rho) = T$, unique up to conjugation.

4. Suppose Γ is finitely generated. Then there is a finite subset $S \subset \Gamma$ such that for any $\mathbb{Z}[1/n!]$-algebra k, if $T : \Gamma \to k$ is a GL_n-pseudocharacter, then T is determined by its values on S.

5. If Γ and k are taken to be topological, then the above statements hold in topological/continuous form.

Taylor used GL_n-pseudocharacters to construct Galois representations having certain properties [12, §2].

Recently, V. Lafforgue formulated an analog of GL_n-pseudocharacters which works with GL_n replaced by any connected reductive group H. However, instead of consisting of one function $\Gamma \to k$ satisfying a finite number of relations, these "pseudocharacters" consist of an infinite sequence of algebra morphisms satisfying certain properties. These sequences of morphisms are essentially equivalent to specifying an infinite number of functions $\Gamma^I \to k$, with I ranging over all finite sets, satisfying an infinite number of relations.

Lafforgue also shows how to derive Taylor's result from the above theorem [5, Remark 11.8], using results of Procesi [9] which state that the trace function "generates" all of the algebras $k[GL_n^I]^{\mathrm{Ad}GL_n}$ (where $\mathrm{Ad}GL_n$ denotes the diagonal conjugation action) and which explicitly describe all of the relations between these trace functions.

In Section 2 of this paper, we reformulate Lafforgue's result in terms of a new algebraic structure called an FFG-algebra. Collections of morphisms Ξ_n as above are recast as morphisms between certain FFG-algebras. We then use the finiteness theorems of classical invariant theory to show that, for any H as above, the FFG-algebra derived from the invariants of H is finitely presented. Hence it is always possible to define H-pseudocharacters consisting of finitely many functions $\Gamma^I \to k$ satisfying finitely many relations.

In Section 3, we use invariant theoretic-results of Procesi and others [9, 1, 11] to give explicit finite presentations for the FFG-algebras corresponding to the general and ordinary orthogonal groups GO_n and O_n, the general and ordinary symplectic groups GSp_n and Sp_n, and the special orthonal group SO_n. By extension, we define explicit pseudocharacters for these groups.

Finally, in Section 4, we use our pseudocharacters to investigate the problem of conjugacy vs. element-conjugacy for represention $\Gamma \to H$, when H is a linear algebraic group for which one can define pseudocharacters. We formulate a general sufficient condition for when element-conjugacy implies conjugacy in terms of FFG-algebras, and we conjecture that this condition is also necessary (Conjecture 4.4). We then use our explicit pseudocharacters for $GO_n(\mathbb{C})$, $O_n(\mathbb{C})$, $GSp_{2n}(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ to prove that for any group Γ, semisimple element-conjugate representations from Γ to one of those groups are automatically conjugate. Previous results of this form were only known for $O_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$, and only for compact Γ. We also give a counterexample to the corresponding claim for $SO_{2n}(\mathbb{C})$ ($n \geq 3$) which is simpler than that used in [6, Proposition 3.8], and which extends that result to $SO_6(\mathbb{C})$.

2 General Results on Pseudocharacters

In this section, we define FFG-algebras and use them to reformulate V. Lafforgue's result. Section 2.1 introduces FFG-algebras and the closely related FI-algebras and FFS-algebras, modeled after the FI-modules defined in [2]. Section 2.2 restates V. Lafforgue's result in terms of morphisms between particular FFG-algebras. Finally, section 2.3 shows that the FFG-algebra $k[H^\bullet]^{\mathrm{Ad}H}$ (see Example 2.5) appearing in Theorem 2.12 is "finitely presented" as an FFG-algebra, in an appropriate sense (in fact, it is finitely presented even as an FI-algebra), and we explain how this finite presentation implies the general existence of "finite" pseudocharacters.

2.1 FI-, FFS-, and FFG-algebras.

We denote by FI the category of finite sets, FFS the category of free finitely generated semigroups, and FFG the category of free finitely generated groups. For every finite (nonempty) set I, let $\mathrm{FS}(I)$ (resp. $\mathrm{FG}(I)$) denote the free semigroup (resp. group) generated by I.

The following lemma is easy.

Lemma 2.1. *The category FFS is generated by the following two types of morphisms:*

- *morphisms* $\mathrm{FS}(I) \to \mathrm{FS}(J)$ *that sends generators to generators, i.e., those induced by maps between finite sets* $I \to J$*;*
- *morphisms*

$$\mathrm{FS}(\{x_1, \dots, x_n\}) \to \mathrm{FS}(\{y_1, \dots, y_{n+1}\}), \qquad x_i \mapsto y_i (i < n), x_n \mapsto y_n y_{n+1}.$$

The category FFG is generated by the above two types of morphisms (with FS *replaced by* FG*) together with:*

- *morphisms*

$$\mathrm{FG}(\{x_1, \dots, x_n\}) \to \mathrm{FG}(\{y_1, \dots, y_n\}), \qquad x_i \mapsto y_i (i < n), x_n \mapsto y_n^{-1}.$$

Definition 2.2. Fix a commutative ring k. An FI-algebra (resp. FFS-algebra, FFG-algebra) is a functor from FI (resp. FFS, FFG) to the category of k-algebras. Morphisms between FI-algebras (resp. FFS-algebras, FFG-algebras) are natural transformations of functors.

If $A^\bullet$ is an FI-algebra (resp. FFS-algebra, FFG-algebra) and I is a finite set, we will use A^I to denote the k-algebra corresponding to I under $A^\bullet$, and similary for morphisms $\Theta^\bullet : A^\bullet \to B^\bullet$. If $\phi : I \to J$ (resp. $\mathrm{FS}(I) \to \mathrm{FS}(J)$, $\mathrm{FG}(I) \to \mathrm{FG}(J)$) is a morphism, then we will use A^ϕ to denote the corresponding k-algebra morphism $A^I \to A^J$.

We can define kernels, cokernels, subobjects, quotients, and tensor products over k in the category of FI-algebras (resp. FFS-algebras, FFG-algebras) by using the analogous constructions in the category of k-algebras, applying those constructions to each k-algebra in the image of an FI-algebra. We say that a morphism $\Theta^\bullet$ is injective (resp. surjective, bijective) if each Θ^I has that property.

Remark 2.3. Any FFG-algebra is naturally an FFS-algebra, and any FFS-algebra is naturally an FI-algebra. A morphism of FFG-algebras is also a morphism of FFS-algebras, and a morphism of FFS-algebras is also a morphism of FI-algebras.

Example 2.4. Let Γ be a group and R be a k-algebra. We define an FFG-algebra $\mathrm{Map}(\Gamma^\bullet, R)$ as follows. To the finite set I, we associate $\mathrm{Map}(\Gamma^I, R)$. Next, recall that for any finite set I, $\Gamma^I = \mathrm{Hom}(\mathrm{FG}(I), \Gamma)$. Thus for any group homomorphism $\phi : \mathrm{FG}(I) \to \mathrm{FG}(J)$, we have a natural set map $\Gamma^J \to \Gamma^I$, which induces a k-algebra morphism $\mathrm{Map}(\Gamma^I, R) \to \mathrm{Map}(\Gamma^J, R)$; we associate this morphism to ϕ.

Example 2.5. Let V be an affine variety over k, and let H be a group which acts on V. We define the FI-algebra $k[V^\bullet]^H$ by the association $I \mapsto k[V^I]^H$, where H acts diagonally on V^I. For any set map $\phi : I \to J$, we get a variety map $V^J \to V^I$ defined over k, and this induces a k-algebra morphism $k[V^I]^H \to k[V^J]^H$, which we associate to ϕ. If V is also an algebraic semigroup (resp. group) whose multiplication is compatible with the action of H, then we can similarly give $k[V^\bullet]^H$ a structure of FFS-algebra (resp. FFG-algebra).

For the remainder of this section, we state definitions and claims for FI-algebras, but they easily generalize to FFS-algebras and FFG-algebras.

Definition 2.6. Let $A^\bullet$ be an FI-algebra. Given a subset $\Sigma \subset \sqcup_I A^I$, the FI-algebra span $\mathrm{span}_{\mathrm{FI}}(A^\bullet, \Sigma)$ of Σ in $A^\bullet$ is defined to be the minimum sub-FI-algebra of $A^\bullet$ containing each element of Σ. We define an FI-algebra to be finitely generated if it equals the span of some finite set.

There is another way to characterize finite generation, in terms of free FI-algebras. Let m be a nonnegative integer, and let $\mathbf{m} := \{1, 2, \dots, m\}$ denote the typical set of m elements.

Definition 2.7. The free FI-algebra of degree m, denoted $F_{\mathrm{FI}}(m)^\bullet$, is defined by

$$F_{\mathrm{FI}}(m)^I := k[\{x_\psi | \psi \in \mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, I)\}]$$
$$F_{\mathrm{FI}}(m)^\phi := (x_\psi \mapsto x_{\phi\circ\psi})$$

In the case of FFS-algebras (resp. FFG-algebras), we replace $\mathrm{Hom}_{\mathrm{FI}}(\mathbf{m}, I)$ with $\mathrm{Hom}_{\mathrm{FFS}}(\mathrm{FS}(\mathbf{m}), \mathrm{FS}(I))$ (resp. $\mathrm{Hom}_{\mathrm{FFG}}(\mathrm{FG}(\mathbf{m}), \mathrm{FG}(I))$).

If $A^\bullet$ is an FI-algebra and $a \in A^{\mathbf{m}}$, then it is easy to see that $x_{id_{\mathbf{m}}} \mapsto a$ extends to a unique map of FI-algebras $F_{\mathrm{FI}}(m)^\bullet \to A^\bullet$, and its image is precisely $\mathrm{span}_{\mathrm{FI}}(A^\bullet, a)$. Thus:

Proposition 2.8. *An FI-algebra $A^\bullet$ is finitely generated iff it admits a surjective morphism $\bigotimes_i F_{\mathrm{FI}}(m_i) \to A^\bullet$ for some finite sequence of integers (m_i).*

Definition 2.9. Let $A^\bullet$ be an FI-algebra. An FI-ideal of $A^\bullet$ is an association $\mathfrak{a}$ taking each finite set I to an ideal $\mathfrak{a}^I$ of A^I, such that for all morphisms $\phi \in \mathrm{Hom}_{\mathrm{FI}}(I, J)$, we have $A^\phi(\mathfrak{a}^I) \subset \mathfrak{a}^J$. Given a morphism of FI-algebras $\Theta^\bullet : A^\bullet \to B^\bullet$, we define the kernel of $\Theta^\bullet$ to be the association $\ker(\Theta^\bullet)$ taking each finite set I to the ideal $\ker(\Theta^I : A^I \to B^I)$ of A^I. We define the radical of an FI-ideal $\mathfrak{a}$ to be the association $I \mapsto \sqrt{\mathfrak{a}^I}$, where the radical is taken in A^I.

The following lemma is easy.

Lemma 2.10. *(i) Let $\Theta^\bullet : A^\bullet \to B^\bullet$ be a morphism of FI-algebras. Then $\ker(\Theta^\bullet)$ is an FI-ideal of $A^\bullet$.*

(ii) Let $\mathfrak{a}$ be an FI-ideal of $A^\bullet$. Then there exists an FI-algebra $B^\bullet$ and a surjective morphism $\Theta^\bullet : A^\bullet \to B^\bullet$ such that $\ker(\Theta^\bullet) = \mathfrak{a}$. Furthermore, the pair $(B^\bullet, \Theta^\bullet)$ is unique up to unique isomorphism.

We denote the FI-algebra in part (ii) by $A^\bullet/\mathfrak{a}$.

Definition 2.11. Let $A^\bullet$ be an FI-algebra. Given a subset $\Sigma \subset \sqcup_I A^I$, we define the FI-ideal generated by Σ to be the minimum FI-ideal of $A^\bullet$ containing each element of Σ. We define an FI-ideal to be finitely generated if it is generated by some finite set.

2.2 Lafforgue's theorem.

Let H be a connected reductive group defined over k, and let Γ be an abstract group. For any finite set I, we let $\mathrm{Ad}H$ denote the diagonal conjugation action of H on H^I, and we let $k[H^\bullet]^{\mathrm{Ad}H}$ denote the FFG-algebra in Example 2.5 corresponding to this action. Also let $\mathrm{Map}(\Gamma^\bullet/\mathrm{Ad}\Gamma, k)$ be the sub-FFG-algebra of $\mathrm{Map}(\Gamma^\bullet, k)$ consisting of functions with are invariant under diagonal conjugation by Γ. Then we can rephrase V. Lafforgue's result as follows.

Theorem 2.12 ([5, Proposition 11.7])**.** *Let H be a connected reductive group defined over k. Assume* $\mathrm{char}\, k = 0$. *Then:*

(i) There is a natural bijection between

$$\{H(\overline{k})\text{-conjugacy classes of semisimple representations } \rho : \Gamma \to H(\overline{k})\}$$

and FFS-*algebra morphisms*

$$\Theta^\bullet : \overline{k}[H^\bullet]^{\mathrm{Ad}H} \to \mathrm{Map}(\Gamma^\bullet/\mathrm{Ad}\Gamma, \overline{k}).$$

The bijection is given by sending $\rho : \Gamma \to H(\overline{k})$ to the FFS-*algebra morphism $\Theta^\bullet$ given by*

$$\Theta^{\mathbf{n}}(f)(\gamma_1, \dots, \gamma_n) = f(\rho(\gamma_1), \dots, \rho(\gamma_n)).$$

(ii) If $\Theta^\bullet$ restricts to give an FFS-*algebra morphism $k[H^\bullet]^{\mathrm{Ad}H} \to \mathrm{Map}(\Gamma^\bullet/\mathrm{Ad}\Gamma, k)$, then the corresponding conjugacy class contains a representation $\rho : \Gamma \to H(k')$ for some finite extension k'/k.*

(iii) If Γ is profinite, H is split over k, and k is a finite extension of $\mathbb{Q}_l$ for some l, then (i) and (ii) hold with "representation" replaced by "continuous representation" and with $\mathrm{Map}(\Gamma^\bullet/\mathrm{Ad}\Gamma, \overline{k})$ replaced by the FFG-*algebra $C(\Gamma^\bullet/\mathrm{Ad}\Gamma, \overline{k})$ of continuous $\mathrm{Ad}\Gamma$-invariant maps $\Gamma^I \to \overline{k}$ (and similarly for $\mathrm{Map}(\Gamma^\bullet/\mathrm{Ad}\Gamma, k)$).*

Corollary 2.13. *The above theorem holds with* FFS-*algebra morphisms replaced by* FFG-*morphisms.*

Proof. Any FFG-algebra morphism $\overline{k}[H^\bullet]^{\mathrm{Ad}H} \to \mathrm{Map}(\Gamma^\bullet/\mathrm{Ad}\Gamma, \overline{k})$ is also an FFS-algebra morphism. Conversely, given an FFS-algebra morphism $\Theta^\bullet : \overline{k}[H^\bullet]^{\mathrm{Ad}H} \to \mathrm{Map}(\Gamma^\bullet/\mathrm{Ad}\Gamma, \overline{k})$, we get a representation $\rho : \Gamma \to H(\overline{k})$ by the theorem, and then the relation

$$\Theta^{\mathbf{n}}(f)(\gamma_1, \dots, \gamma_n) = f(\rho(\gamma_1), \dots, \rho(\gamma_n))$$

shows that $\Theta^\bullet$ is in fact an FFG-algebra morphism. $\square$

2.3 Explicit descriptions of pseudocharacters.

In this section, we will show that whenever Lafforgue's theorem applies to H, the FFG-algebra $k[H^\bullet]^{\mathrm{Ad}H}$ is "finitely presented" in an appropriate sense. In fact, this is true even of $k[H^\bullet]^{\mathrm{Ad}H}$ as an FI-algebra. As a consequence, it is always possible to define pseudocharacters for H very explicitly, in a sense which will be made clear in Section 3.

Theorem 2.14. *Assume k is a field of characteristic 0. Let H be a reductive group over k which acts linearly on a finite-dimensional k-vector space V. Then the FI-algebra $k[V^\bullet]^H$ is finitely generated.*

Proof. By the Hilbert-Nagata theorem, for every finite set I, $k[V^I]^H$ is finitely generated as a k-algebra. Let $d = \dim V$, and let Ω be a finite set of multihomogenous k-algebra generators for $k[V^d]^H$. Then by [10, Theorem 11.1.1.1], for all n, $k[V^n]^H$ is generated by polarizations of elements of $k[V^d]^H$, from which one can see that $k[V^n]^H$ is generated by polarizations of elements of Ω. In other words, $\varinjlim_n k[V^n]^H$ is generated by polarizations of elements of Ω.

Now easily any polarization of a multihomogeneous function h can be obtained as a further polarization of any full polarization of h (up to a scalar multiple), since $\operatorname{char} k = 0$. So, letting Σ be a finite set containing one full polarization of each element of Ω, $\varinjlim_n k[V^n]^H$ is also generated by Σ under polarization.

Now let $f \in k[V^n]^H$ for some n. We claim that f is in the $\mathbf{n}$-part of $\operatorname{span}_{\mathrm{FI}}(k[V^\bullet]^H, \Sigma)$. By the above paragraph, there are elements $g_1, \dots, g_r$ in Σ with polarizations $g_1^1, \dots, g_1^{i_1}, \dots, g_r^1, \dots, g_r^{i_r}$ such that f is in the k-algebra generated by the g_j^l. Since any further polarization of a full polarization results from vector variable substitutions in the full polarization, we see that each g_j^l is in the FI-algebra span of Σ. Using the natural embeddings $k[V^s]^H \subset k[V^t]^H$ whenever $s \le t$, which correspond to the natural embeddings $\mathbf{s} \subset \mathbf{t}$, we can assume that all g_j^l are in $k[V^N]^H$ for some N. Then the image of f under the embedding $k[V^n]^H \subset k[V^N]^H$ lies in the k-algebra generated by the g_j^l. Mapping this image of f back to $k[V^n]^H$ using the map $(k[V^\bullet]^H)^\phi$ for some $\phi : \mathbf{N} \to \mathbf{n}$ which is the identity on $\mathbf{n}$, we thus find that f is in the FI-algebra span of Σ. $\square$

Corollary 2.15. *Assume k is a field of characteristic 0. Let H be a reductive linear algebraic group over k. Then the* FFG-*algebra* $k[H^\bullet]^{\mathrm{Ad}H}$ *is finitely generated as an* FI-*algebra, hence also as an* FFS- *and* FFG-*algebra.*

Proof. Let d be such that H is an affine sub-group variety of GL_d over k. Using the embedding of GL_d into $M_d \times \mathbb{A}^1$ given by sending $A \in GL_d(k)$ to $(A, \det(A)^{-1})$, we can consider GL_d, hence H, as an affine sub-monoid variety of $M_d \times \mathbb{A}^1$. Then H acts linearly on $M_d(k) \times \mathbb{A}^1(k)$ (as a k-vector space) by conjugation on $M_d(k)$; call this action $\mathrm{Ad}H$. Obviously this action restricts to give the conjugation action $\mathrm{Ad}H$ of H on itself.

By the theorem, $k[M_d \times \mathbb{A}^1]^{\mathrm{Ad}H}$ is finitely generated as an FI-algebra. Now let $\mathfrak{a} \subset k[M_d \times \mathbb{A}^1]$ be the ideal which cuts out H as a variety. Then by [4, Lemma 2.2.1 and Corollary 2.4.5] , for all finite sets I,

$$k[H^I]^{\mathrm{Ad}H} = \frac{k[(M_d \times \mathbb{A}^1)^I]^{\mathrm{Ad}H}}{\mathfrak{a}^I \cap k[(M_d \times \mathbb{A}^1)^I]^{\mathrm{Ad}H}}.$$

Hence we can exhibit $k[H^\bullet]^{\mathrm{Ad}H}$ as a quotient of $k[(M_d \times \mathbb{A}^1)^\bullet]^{\mathrm{Ad}H}$, and obviously any quotient of a finitely generated FI-algebra is finitely generated. $\square$

Recall that if $A^\bullet$ is finitely generated as an FI-algebra, then there is a surjective morphism $\bigotimes_i F_{\mathrm{FI}}(m_i) \to A^\bullet$ for some finite sequence of integers (m_i). We now show that the kernel of such a morphism is always finitely generated as an FI-ideal, hence any finitely generated FI-algebra is actually finitely presented. To do this, we prove a statement analogous to the Noetherian property of polynomial rings over k, as follows.

Proposition 2.16. *Let (m_i) be a finite sequence of integers, and let $A^\bullet = \otimes_i F_{\mathrm{FI}}(m_i)$. Let $\mathfrak{a}$ be an FI-ideal of $A^\bullet$. Then $\mathfrak{a}$ is finitely generated.*

Proof. Let $M = \max\{m_i\}$. For fixed i, let $\iota : \mathbf{m_i} \to \mathbf{M}$ be the canonical injection, and define a morphism $\Theta^\bullet : F_{\mathrm{FI}}(M) \to F_{\mathrm{FI}}(m_i)$ by sending $x_{id_{\mathbf{M}}}$ to x_ι. Let $\pi : \mathbf{M} \to \mathbf{m_i}$ be some map such that $\pi \circ \iota = id_{\mathbf{m_i}}$. Then $x_{id_{\mathbf{m_i}}} = (F_{\mathrm{FI}}(m_i))^\pi(x_\iota)$, so $\Theta^\bullet$ is surjective. Then for any ideal $\mathfrak{b}$ of $F_{\mathrm{FI}}(m_i)$, we can define an FI-ideal $(\Theta^\bullet)^{-1}(\mathfrak{b})$ of $F_{\mathrm{FI}}(M)$ by setting $\left((\Theta^\bullet)^{-1}(\mathfrak{b})\right)^I = (\Theta^I)^{-1}(\mathfrak{b}^I)$. Easily if $(\Theta^\bullet)^{-1}(\mathfrak{b})$ is finitely generated, then so is $\mathfrak{b}$.

Hence WLOG all $m_i = M$ for some integer M. Then $A^\bullet = F_{\mathrm{FI}}(M)^{\otimes n}$ for some integer n. Now let $\mathfrak{b}$ be the FI-ideal of $A^\bullet$ generated by $\mathfrak{a}^{\mathbf{M}}$. Then the identity maps $(A^\bullet/\mathfrak{a})^{\mathbf{M}} \to (A^\bullet/\mathfrak{b})^{\mathbf{M}}$, $(A^\bullet/\mathfrak{b})^{\mathbf{M}} \to (A^\bullet/\mathfrak{a})^{\mathbf{M}}$ induce maps $A^\bullet/\mathfrak{a} \to A^\bullet/\mathfrak{b}$, $A^\bullet/\mathfrak{b} \to A^\bullet/\mathfrak{a}$ which are inverses to each other and which commute with the identity map on $A^\bullet$, by properties of free FI-algebras. Hence $\mathfrak{a} = \mathfrak{b}$, so $\mathfrak{a}$ is generated by $\mathfrak{a}^{\mathbf{M}}$ as an FI-ideal.

Now from the definition, we see that $A^{\mathbf{M}}$ is a finitely generated k-algebra, hence is Noetherian. Thus $\mathfrak{a}^{\mathbf{M}}$ is finitely generated as an ideal in $A^{\mathbf{M}}$. Any finite set of generators then provides a finite set of generators for $\mathfrak{a}$. $\square$

Now for fixed H, by choosing a finite set of generators for $k[H^\bullet]^{\mathrm{Ad}H}$ as an FFG-algebra, as well as a finite set of generators for the FFG-ideal of relations between those generators (or even a set of generators up to radical), we can define pseudocharacters for H in terms of a finite set of functions satisfying finitely many relations. This technique was first demonstrated in [5, Remark 11.8], wherein V. Lafforgue implicitly gives a finite presentation for $k[GL_n^\bullet]^{\mathrm{Ad}GL_n}$ and explains how it implies Taylor's original result on GL_n-pseudocharacters. We further illustrate the technique with examples in Section 3 below.

3 Explicit Pseudocharacters for Classical Groups

3.1 (General) Orthogonal Group

We now present new results which establish pseudocharacters for the orthogonal and general orthogonal groups. Assume k is an algebraically closed field of characteristic 0.

Let $GO_n(k) = \{A \in M_n(k) \mid \text{for some } \lambda \in k^\times,\ AA^t = \lambda I\}$ be the n-dimensional general orthogonal group. It is a connected reductive algebraic group, and it is in fact an affine subvariety of GL_n, hence of $M_n \times \mathbb{A}^1$. Define a function $\lambda : GO_n(k) \to k$ by $AA^t = \lambda(A)I$. Then $\lambda \in k[GO_n]^{\mathrm{Ad}GO_n}$.

Proposition 3.1. *$k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ is generated as an FFG-algebra by* tr *and* λ.

Proof. Since $GO_n(k) = k^\times \cdot O_n(k)$, $k[M_n^\bullet]^{\mathrm{Ad}GO_n} = k[M_n^\bullet]^{\mathrm{Ad}O_n}$. By Procesi's results on the invariants of the orthogonal group acting on matrices by conjugation [9, Theorem 7.1], for all m, $k[M_n^m]^{\mathrm{Ad}O_n}$ is generated as a k-algebra by invariants $\mathrm{tr}(M)$, where $M \in \mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$. Then $k[(M_n \times \mathbb{A}^1)^m]^{\mathrm{Ad}O_n}$ is generated as a k-algebra by the $\mathrm{tr}(M)$ and by the coordinate functions for the m copies of $\mathbb{A}^1$, which we will denote $\det^{-1}(A_1), \ldots, \det^{-1}(A_m)$. By [4, Lemma 2.2.1 and Corollary 2.4.5] , $k[GO_n^m]^{\mathrm{Ad}GO_n}$ is a quotient of $k[(M_n \times \mathbb{A}^1)^m]^{\mathrm{Ad}GO_n}$ for all m, so it is also generated by the invariants $\mathrm{tr}(M)$ and $\det^{-1}(A_i)$. Then using the identity $A^t = \lambda(A)A^{-1}$ for $A \in GO_n(k)$, we see that any invariant $\mathrm{tr}(M)$ is in the FFG-algebra generated by tr and λ. Also, using the identity $\det^{-1}(A) = \det(A^{-1})$ and the fact that we can express $\det(A^{-1})$ in terms of $\mathrm{tr}(A^{-1}), \dots, \mathrm{tr}(A^{-n})$, we see that any invariant $\det^{-1}(A_i)$ is in the FFG-algebra generated by tr. $\square$

The relations between the invariants are more complicated to describe. We first summarize Procesi's result on relations between the generators $\mathrm{tr}(M)$ for $k[M_n^m]^{\mathrm{Ad}O_n}$.

Let R be the polynomial ring over k with indeterminates T_M as M varies over $\mathrm{FS}(\{A_1, A_1^t, \dots, A_m, A_m^t\})$, except that we make the identifications $T_{MN} = T_{NM}$ and $T_M = T_{M^t}$ for all words M and N (where M^t is defined in the obvious way). Let $\pi : R \to k[M_n^m]^{\mathrm{Ad}O_n}$ be the k-algebra homomorphism sending each T_M to $\mathrm{tr}(M)$, which by [9, Theorem 7.1] is surjective.

Given $M_1, M_2, \dots, M_{n+1} \in \mathrm{FS}(\{A_1, A_1^t, \dots, A_m, A_m^t\})$ and an integer $0 \le j \le (n+1)/2$, define $F_{j,n+1}(M_1, M_2, \dots, M_{n+1}) \in R$ as follows. Let s be given by $n+1 = 2j+s$. Let S be a set of formal symbols (a,b), where each a and b is one of the formal symbols $u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1}$. Let $D^j = \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) D^j_\sigma$ be the following $(n+1) \times (n+1)$ determinant, as a function of symbols in S:

$$\begin{vmatrix}
(u_1, u_{j+s+1}) & (u_1, u_{j+s+2}) & \cdots & (u_1, u_{n+1}) & (u_1, v_{j+1}) & (u_1, v_{j+2}) & \cdots & (u_1, v_{n+1}) \\
\vdots & & & & & & & \\
(u_{j+s}, u_{j+s+1}) & (u_{j+s}, u_{j+s+2}) & \cdots & (u_{j+s}, u_{n+1}) & (u_{j+s}, v_{j+1}) & (u_{j+s}, v_{j+2}) & \cdots & (u_{j+s}, v_{n+1}) \\
(v_1, u_{j+s+1}) & (v_1, u_{j+s+2}) & \cdots & (v_1, u_{n+1}) & (v_1, v_{j+1}) & (v_1, v_{j+2}) & \cdots & (v_1, v_{n+1}) \\
\vdots & & & & & & & \\
(v_j, u_{j+s+1}) & (v_j, u_{j+s+2}) & \cdots & (v_j, u_{n+1}) & (v_j, v_{j+1}) & (v_j, v_{j+2}) & \cdots & (v_j, v_{n+1})
\end{vmatrix}$$

Next, using the formal identities $(a,b) = (b,a)$ and allowing the symbols (a,b) to commute with each other, write each monomial D^j_σ of D^j in the form

$$D^j_\sigma = (w_{i_1}, \bar{w}_{i_2})(w_{i_2}, \bar{w}_{i_3}) \dots (w_{i_j}, \bar{w}_{i_1}) \cdot (w_{j_1}, \bar{w}_{j_2})(w_{j_2}, \bar{w}_{j_3}) \dots (w_{j_s}, \bar{w}_{j_1}) \dots$$

where w_a stands for either u_a or v_a, and by definition, $\bar{u}_a = v_a$ and $\bar{v}_a = u_a$. Now define $T^j_\sigma(M_1, \dots, M_{n+1})$ by

$$T^j_\sigma(M_1, \dots, M_{n+1}) = T_{N_{i_1} N_{i_2} \dots N_{i_j}} T_{N_{j_1} N_{j_2} \dots N_{j_j}} \dots,$$

where $N_a = M_a$ or $N_a = M_a^t$, according to the inductively defined rules:

- $N_{i_1} = M_{i_1}$, if $w_{i_1} = v_{i_1}$; else $N_{i_1} = M_{i_1}^{-1}$.
- Set $N_{i_{t+1}}$ to be the same type as N_{i_t} (transposed or not transposed) if and only if w_{i_t} and $w_{i_{t+1}}$ stand for instances of the same letter (u or v).

Then

$$F_{j,n+1}(M_1, \dots, M_{n+1}) := \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) T^j_\sigma(M_1, \dots, M_{n+1})$$

is the result of replacing each D^j_σ with T^j_σ in D^j. Because $T_{MN} = T_{NM}$ and $T_M = T_{M^t}$ by assumption, the functions T^j_σ are well-defined, hence so is $F_{j,n+1}$. Note that $F_{0,n+1}(M_1, \dots, M_{n+1})$ reduces to the non-trivial relation for GL_n-pseudocharacters.

Theorem 3.2 ([9, Theorem 8.4(a)]). $\ker(\pi)$ *is the ideal of* R *generated by the* $F_{j,n+1}(M_1,\dots,M_{n+1})$, $0 \le j \le (n+1)/2$, *as the* M_i *vary over* $\mathrm{FS}(\{A_1, A_1^t, \dots, A_m, A_m^t\})$.

Now let ψ be the composition of π with the map $k[M_n^m]^{\mathrm{Ad}GO_n} \hookrightarrow k[(M_n \times \mathbb{A}^1)^m]^{\mathrm{Ad}GO_n} \twoheadrightarrow k[GO_n^m]^{\mathrm{Ad}GO_n}$, which is still surjective by the proof of Proposition 3.1. Intuitively, one should expect $\ker(\psi)$ to be $\ker(\pi)$ plus the relations of the form $T_{MNN^tP} = T_{MP}T_{NN^t}$, since GO_n is defined by the condition that NN^t is a scalar matrix for all $N \in GO_n(k)$. The next proposition shows that this is indeed the case, at least up to radical.

Proposition 3.3. $\ker(\psi)$ *is the radical of the ideal generated by* $\ker(\pi)$ *and the relations* $T_{MNN^tP} - T_{MP}T_{NN^t}$ *as* M, N, P *vary over all semigroup words in the letters* A_i, A_i^t.

Proof. Let $J \subset R$ be the ideal generated by $\ker(\pi)$ and the $T_{MNN^tP} - T_{MP}T_{NN^t}$. It suffices to prove that $\pi(\ker(\psi)) = \sqrt{\pi(J)}$. Now $\sqrt{\pi(\ker(\psi))} = \pi(\ker(\psi))$ because $k[M_n^m]^{\mathrm{Ad}GO_n}/\pi(\ker(\psi)) \cong k[GO_n^m]^{\mathrm{Ad}GO_n}$ is reduced, so by the Nullstellensatz, it suffices to prove that $\pi(\ker(\psi))$ and $\pi(J)$ define the same subvariety of $\mathrm{Spec}(k[M_n^m]^{\mathrm{Ad}GO_n})$. Using the $\mathrm{tr}(M)$ as coordinate functions for $\mathrm{Spec}(k[M_n^m]^{\mathrm{Ad}GO_n})$, the subvariety associated to $\pi(\ker(\psi))$ is the set of all points of the form $(\mathrm{tr}(M(A_i \mapsto B_i)))_{M\in\{A_1,A_1^t,\dots,A_m,A_m^t\}}$ for some $B_1,\dots,B_m \in GO_n(k)$, while the subvariety associated to $\pi(J)$ is the set of all points of the form $(\mathrm{tr}(M(A_i \mapsto C_i)))_{M\in\{A_1,A_1^t,\dots,A_m,A_m^t\}}$ for some $C_1,\dots,C_m \in M_n(k)$ such that $\mathrm{tr}(MNN^tP) = \mathrm{tr}(MP)\mathrm{tr}(NN^t)$ whenever M, N, and P are semigroup words in the C_i and C_i^t. The following lemma shows that these two subvarieties are equal, proving the claim. ☐

Lemma 3.4. *Let* $C_1,\dots,C_m \in M_n(k)$ *be such that* $\mathrm{tr}(MNN^tP) = \mathrm{tr}(MP)\mathrm{tr}(NN^t)$ *whenever* M, N, *and* P *are semigroup words in the* C_i *and* C_i^t. *Then there exist* $B_1,\dots,B_m \in GO_n(k)$ *such that for all* $M \in \mathrm{FS}(\{A_1, A_1^t, \dots, A_m, A_m^t\})$, $\mathrm{tr}(M(A_i \mapsto B_i)) = \mathrm{tr}(M(A_i \mapsto C_i))$.

Proof. Let (V, B) be the bilinear space with $V \cong k^n$ and B the standard nondegenerate symmetric bilinear form, i.e., the dot product. Let A be the noncommutative k-algebra

$$A := k[C_1,\dots,C_r,C_1^t,\dots,C_r^t] \subset M_n(k),$$

which has the natural involution $(-)^t$. Then the natural representation $\rho : A \hookrightarrow M_n(k) \cong \mathrm{End}(V,B)$ is orthogonal, i.e., it preserves involutions.

Then by [9, Theorem 15.2(b)(c)] and the fact that all nondegenerate bilinear forms on V are equivalent, there exists a semisimple orthogonal representation $\rho^{ss} : A \to \mathrm{End}(V)$ such that $\mathrm{tr}(\rho) = \mathrm{tr}(\rho^{ss})$. Thus setting $B_i = \rho^{ss}(C_i)$, we will be done once we prove that $B_i \in GO_n(k)$. Now for any $D \in A$, we have

$$\begin{aligned}\mathrm{tr}((B_iB_i^t - \mathrm{tr}(B_iB_i^t)I)\rho^{ss}(D)) &= \mathrm{tr}(\rho^{ss}((C_iC_i^t - \mathrm{tr}(C_iC_i^t)I)D))\\ &= \mathrm{tr}(\rho((C_iC_i^t - \mathrm{tr}(C_iC_i^t)I)D))\\ &= \mathrm{tr}((C_iC_i^t)D) - \mathrm{tr}(C_iC_i^t)\mathrm{tr}(D)\\ &= 0\end{aligned}$$

by assumption. Since ρ^{ss} is semisimple, tr is a nondegenerate bilinear form on $\mathrm{Im}(\rho^{ss})$, so this shows that $B_iB_i^t - \mathrm{tr}(B_iB_i^t)I = 0$, proving the claim. ☐

Next, let S be the polynomial ring over k with indeterminates:

- U_M as M varies over words in $\mathrm{FG}(\{A_1, \ldots, A_m\})$, with the identifications $U_1 = n$ and $U_{MN} = U_{NM}$ for all words M, N
- l_M as M varies over words in $\mathrm{FG}(\{A_1, \ldots, A_m\})$, with the identifications $l_1 = 1$ and $l_{MN} = l_M l_N$ for all words M, N,

and let $L \subset S$ be the ideal generated by relations of the form $U_M - \lambda_M U_{M^{-1}}$. Then we have a surjective map $\rho : S/J \twoheadrightarrow k[GO_n^m]^{\mathrm{Ad}GO_n}$ defined by $\rho(U_M) = \mathrm{tr}(M)$ and $\rho(l_M) = \lambda(M)$.

Now easily ψ factors through ρ via the map $\tau : R \to S/J$ which sends T_M to $l_{M'} U_{M''}$, where M' is the product (with multiplicity) of all letters $A_1, \ldots, A_n$ which appear transposed in M, and M'' is the result of substituting all transposed letters A_i^t in M with A_i^{-1}. From this and the above proposition, noting that $\tau(T_{MNN^tP} - T_{MP}T_{NN^t}) = 0$, we get:

Corollary 3.5. $\ker(\rho : S \to k[GO_n^m]^{\mathrm{Ad}GO_n})$ *is the radical of the ideal ideal generated by the relations:*

- $U_M - \lambda_M U_{M^{-1}}$ *as M varies over words in* $\mathrm{FG}(\{A_1, \ldots, A_m\})$
- $G_{j,n+1}(M_1, \ldots, M_{n+1})$, $0 \leq j \leq (n+1)/2$, *as the M_i vary over words in* $\mathrm{FS}(\{A_1, A_1^t, \ldots, A_m, A_m^t\})$. *Here $G_{j,n+1}(M_1, \ldots, M_{n+1})$ is the same as $F_{j,n+1}(M_1, \ldots, M_{n+1})$ except that we replace each T_M with $l_{M'} U_{M''}$, where M' is the product (with multiplicity) of all letters $A_1, \ldots, A_n$ which appear transposed in M, and $M'' \in \mathrm{FG}(\{A_1, \ldots, A_m\})$ is the result of substituting all transposed letters A_i^t in M with A_i^{-1}.*

Note that $\tau(F_{j,n+1}(M_1, \ldots, M_{n+1}))$ equals the image of $G_{j,n+1}(M_1, \ldots, M_{n+1})$ modulo J.

Finally, we get a finite presentation for $k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ as an FFG-algebra.

Corollary 3.6. *Let $N \in \mathbb{N}$ be such that $k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ is generated by $k[GO_n^N]^{\mathrm{Ad}GO_n}$ as an FI-algebra and $N \geq n+1$. Let $A^\bullet$ be the free FFG-algebra on letters T, l in degree N. Then the FFG-algebra map $\Theta^\bullet : A^\bullet \to k[GO_n^\bullet]^{\mathrm{Ad}GO_n}$ sending T to $\mathrm{tr}(A_1)$ and l and $\lambda(A_1)$ is surjective. Denote the generators of* $\mathrm{FG}(\mathbf{N})$ *by $g_1, \ldots, g_N$, and for $g \in \mathrm{FG}(\mathbf{N})$, let ϕ_g denote some fixed map* $\mathrm{FG}(\mathbf{N}) \to \mathrm{FG}(\mathbf{N})$ *sending g_1 to g. Then the kernel of $\Theta^\bullet$ is the radical of the FFG-algebra ideal generated by the relations:*

- $A^{\phi_1}(T) - n$
- $A^{\phi_{g_1g_2}}(T) - A^{\phi_{g_2g_1}}(T)$
- $A^{\phi_1}(l) - 1$
- $A^{\phi_{g_1g_2}}(T) - A^{\phi_{g_1}}(T)A^{\phi_{g_2}}(T)$
- $T - lA^{\phi_{g_1^{-1}}}(T)$
- $G_{j,n+1}(g_1, \ldots, g_{n+1})$, $0 \leq j \leq (n+1)/2$, *which is defined in the same way as* $G_{j,n+1}(M_1, \ldots, M_{n+1})$ *by abuse of notation, except that we replace each symbol l_g with $A^{\phi_g}(l)$ and each T_g with $A^{\phi_g}(T)$.*

We now apply Lafforgue's result.

Definition 3.7. Let Γ be a group. A GO_n-pseudocharacter of Γ over k is a pair (T, l), consisting of a set map $T : \Gamma \to k$ and a group homomorphism $l : \Gamma \to k^\times$, such that:

- $T(1) = n$
- For all $\gamma_1, \gamma_2 \in \Gamma$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$
- For all $\gamma \in \Gamma$, $T(\gamma) = l(\gamma)T(\gamma^{-1})$
- For all integers $0 \le j \le (n+1)/2$ and for all $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$, T and l satisfy the relation $H_{j,n+1}(l, T, \gamma_1, \dots, \gamma_{n+1}) = 0$, where $H_{j,n+1}(l, T, \gamma_1, \dots, \gamma_{n+1})$ is defined in the same way as $G_{j,n+1}(\gamma_1, \dots, \gamma_{n+1})$ by abuse of notation, except that we replace each symbol l_γ with $l(\gamma)$ and each T_γ with $T(\gamma)$.

Definition 3.8. An O_n-pseudocharacter of Γ over k is a set map $T : \Gamma \to k$ such that $(T, 1)$ is a GO_n-pseudocharacter.

Theorem 3.9. *(i) There is a natural bijection between $GO_n(k)$-conjugacy classes of semisimple representations $\rho : \Gamma \to GO_n(k)$ and GO_n-pseudocharacters (T, l) of Γ over k. The bijection is given by sending $\rho : \Gamma \to GO_n(k)$ to $(\mathrm{tr}(\rho), \lambda(\rho))$, where $\lambda(\rho)(\gamma) := \lambda(\rho(\gamma))$.*

(ii) There is a natural bijection between $O_n(k)$-conjugacy classes of semisimple representations $\rho : \Gamma \to O_n(k)$ and O_n-pseudocharacters T of Γ over k. The bijection is given by sending $\rho : \Gamma \to O_n(k)$ to $\mathrm{tr}(\rho)$.

(iii) If Γ is profinite and k is a complete extension of $\mathbb{Q}_l$ for some l, then (i) and (ii) hold with "representation" replaced by "continuous representation" and with "GO_n-pseudocharacters" replaced by "continuous GO_n-pseudocharacters". Here (T, l) is called continuous if both T and l are continuous.

Note that we get the above result for $O_n(k)$ even though it is not connected.

Remark 3.10. The above results can also be proven by modifying Taylor's proof for GL_n-pseudocharacters [12, Theorem 1]. In fact, one can generalize the above result to algebras, as follows. First define a $*$-algebra to be a (possibly noncommutative) k-algebra with an involution $*$. Define an orthogonal n-dimensional representation of a $*$-algebra R to be a k-algebra morphism $R \to M_n(k)$ mapping $*$ to the transpose. Then one can define n-dimensional orthogonal pseudocharacters of a $*$-algebra R similarly to the definition of O_n-pseudocharacters above. Using [9, Theorem 15.3] in place of [12, Lemma 2] in Taylor's proof, one can prove that these are in bijection with $O_n(k)$-conjugacy classes of semisimple orthogonal representations of R. By taking R to be the group algebra $k[\Gamma]$ with involution determined by $(\gamma)^* = l(g)(g^{-1})$ for $\gamma \in \Gamma$, one recovers Theorem 3.9.

3.2 (General) Symplectic Group

Again assume k is an algebraically closed field of characteristic 0. Let $GSp_{2n}(k) = \{A \in M_{2n}(k) \mid$ for some $\lambda \in k^\times$, $AA^* = \lambda I\}$ be the n-dimensional general symplectic group; here $*$ is the symplectic involution

$$A^* := \Omega^{-1} A^T \Omega$$

where

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the matrix of the standard symplectic form. It is a connected reductive algebraic group, and it is in fact an affine subvariety of GL_{2n}, hence of $M_{2n} \times \mathbb{A}^1$.

The results and proofs for GSp_{2n} are exactly analogous to those for GO_n, except that instead of starting with the relations $F_{j,n+1}$ defined above, we start with the relations $F^i_{h,n}$, for $1 \leq i \leq n+1$ and $0 \leq h < i$, defined in [9, Theorem 10.2(a)]. For convenience, we state the analog of Theorem 3.9; from this and the original proof, it is easy to read off a finite presentation for $k[GSp^\bullet_{2n}]^{\mathrm{Ad}GSp_{2n}}$ as an FFG-algebra.

Define a function $\lambda : GSp_{2n}(k) \to k$ by $AA^t = \lambda(A)I$. Note that $\lambda \in k[GSp_{2n}]^{\mathrm{Ad}GSp_{2n}}$.

Definition 3.11. Let Γ be a group. A GSp_n-pseudocharacter of Γ over k is a pair (T, l), consisting of a set map $T : \Gamma \to k$ and a group homomorphism $l : \Gamma \to k^\times$, such that:

- $T(1) = n$
- For all $\gamma_1, \gamma_2 \in \Gamma$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$
- For all $\gamma \in \Gamma$, $T(\gamma) = l(\gamma)T(\gamma^{-1})$
- For all integers $1 \leq i \leq n+1$ and $0 \leq h < i$, and for all $\gamma_1, \ldots, \gamma_{n+i} \in \Gamma$, T and l satisfy the relation $H^i_{h,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+i}) = 0$, where $H^i_{h,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+i})$ is defined as follows:
  - Taking $A_1, \ldots, A_{n+i}$ to be matrix variables, define $G^i_{h,n+1}(A_1, \ldots, A_{n+i})$ to be the same as $F^i_{h,n+1}(A_1, \ldots, A_{n+i})$, except that we replace each $\mathrm{tr}(M)$ with formal symbols $l(M')T(M'')$, where M' is the product (with multiplicity) of all letters $A_1, \ldots, A_n$ which appear transposed in M, and $M'' \in \mathrm{FG}(\{A_1, \ldots, A_m\})$ is the result of substituting all transposed letters A^t_i in M with A^{-1}_i.
  - Define $H^i_{h,n+1}(l, T, \gamma_1, \ldots, \gamma_{n+i})$ in the same way as $G^i_{h,n+1}(\gamma_1, \ldots, \gamma_{n+i})$ by abuse of notation, except that we replace each symbol $l(\gamma)$ with its actual value (using the given l), and similarly for each $T(\gamma)$.

Definition 3.12. An Sp_{2n}-pseudocharacter of Γ over k is a set map $T : \Gamma \to k$ such that $(T, 1)$ is a GSp_{2n}-pseudocharacter.

Theorem 3.13. *(i) There is a natural bijection between $GSp_{2n}(k)$-conjugacy classes of semisimple representations $\rho : \Gamma \to GSp_{2n}(k)$ and GSp_{2n}-pseudocharacters (T, l) of Γ over k. The bijection is given by sending $\rho : \Gamma \to GSp_{2n}(k)$ to $(\mathrm{tr}(\rho), \lambda(\rho))$, where $\lambda(\rho)(\gamma) := \lambda(\rho(\gamma))$.*

(ii) There is a natural bijection between $Sp_{2n}(k)$-conjugacy classes of semisimple representations $\rho : \Gamma \to Sp_{2n}(k)$ and Sp_{2n}-pseudocharacters T of Γ over k. The bijection is given by sending $\rho : \Gamma \to Sp_{2n}(k)$ to $\mathrm{tr}(\rho)$.

(iii) If Γ is profinite and k is a complete extension of $\mathbb{Q}_l$ for some l, then (i) and (ii) hold with "representation" replaced by "continuous representation" and with "GSp_{2n}-pseudocharacters" replaced by "continuous GSp_{2n}-pseudocharacters". Here (T, l) is called continuous if both T and l are continuous.

3.3 Special Orthogonal Group

Odd Dimension When the dimension is $2n+1$ for some n, the invariants of M_{2n+1}^m under simultaneous conjugation by SO_{2n+1} are the same as under conjugation by O_{2n+1}, since every orthogonal matrix is ± 1 times a special orthogonal matrix. Then by [4, Lemma 2.2.1 and Corollary 2.4.5],

$$k[SO_{2n+1}^m]^{\mathrm{Ad} SO_{2n+1}} = \frac{k[SO_{2n+1}^m]^{\mathrm{Ad} O_{2n+1}}}{I \cap k[SO_{2n+1}^m]^{\mathrm{Ad} O_{2n+1}}},$$

where I is the ideal of $k[SO_{2n+1}^m]^{\mathrm{Ad} O_{2n+1}}$ generated by the relations $\det(A_i) = 1$ for A_i a coordinate matrix. Thus $k[SO_{2n+1}^\bullet]^{\mathrm{Ad} SO_{2n+1}}$ is also generated by tr as an FFG-algebra. Also, using Hilbert's Nullstellensatz, it is easy to see that for any m, the ideal of relations between the generators of $k[SO_{2n+1}^m]^{\mathrm{Ad} SO_{2n+1}}$ is the radical of the ideal generated by the O_{2n+1} relations and the relations $\det(A_i) = 1$ (as expressed in terms of $\mathrm{tr}(A_i), \ldots, \mathrm{tr}(A_i^{2n+1})$). Hence the relations between tr for $k[SO_{2n+1}^\bullet]^{\mathrm{Ad} SO_{2n+1}}$ are generated, up to radical, by the relations for $k[O_{2n+1}^\bullet]^{\mathrm{Ad} O_{2n+1}}$ and the relation $\det = 1$ expressed in terms of tr.

Definition 3.14. An (odd-dimensional) SO_{2n+1}-pseudocharacter of G over k is an O_{2n+1}-pseudocharacter $T : G \to k$ which additionally satisfies the relation $\det(T)(g) = 1$ for all $g \in G$, where $\det(T)(g)$ is a polynomial in the $T(g^i)$ such that $\det(\mathrm{tr})(B) = \det(B)$ for all matrices B.

Then the usual result holds by Corollary 2.13 and the above discussion.

Even Dimension When the dimension is $2n$ for some n, the invariant theory of SO_{2n} is more complicated. Now $k[M_{2n}^\bullet]^{\mathrm{Ad} SO_{2n}}$ is generated as an FFG-algebra by tr and the n-argument *linearized Pfaffian* pl, defined as the full polarization of the function

$$\widetilde{\mathrm{pf}}(W) := \mathrm{pf}(W - W^t)$$

where pf is the usual Pfaffian [1].

A result due to Rogora [11] allows us to determine the relations between these generators up to radical, as follows.

Lemma 3.15. *The* FFG*-ideal of relations between the generators* tr *and* pl *for* $k[M_{2n}^\bullet]^{\mathrm{Ad} SO_{2n}}$ *is the radical of the* FFG*-ideal generated by the O_{2n}-relations and the relation described in [11, Theorem 3.2].*

Proof. Let R be a polynomial in terms of the given generators (i.e., in terms of their images under the internal morphisms in the free FFG-algebra) which maps to 0 in $k[M_{2n}^\bullet]^{\mathrm{Ad} SO_{2n}}$. Note that conjugating all inputs to R by an element of $O_{2n}(k) \setminus SO_{2n}(k)$ preserves the value of any generator $\mathrm{tr}(M)$ while negating the value of any generator $\mathrm{pl}(M_1, \ldots, M_n)$. Thus conjugating all inputs of any monomial in R sends that monomial to either itself or its negation; we call the monomial "even" in the former case and "odd" in the latter case. Let R_e and R_o be the sums of all even and odd monomials in R, respectively. Then R_e and $-R_0$ are mapped to the same image in $k[M_{2n}^\bullet]^{\mathrm{Ad} SO_{2n}}$. Then conjugating all of their image's inputs by an element of $O_{2n}(k) \setminus SO_{2n}(k)$, we see that R_e and R_0 also map to the same image in $k[M_{2n}^\bullet]^{\mathrm{Ad} SO_{2n}}$. Hence R_e and R_o both map to 0, so that they are both in the FFG-ideal of relations.

It now suffices to show that the even and odd relations are in the given FFG-ideal. If R_e is an even relation, then each of its monomials consists of traces and of pairs of linearized Pfaffians.

After replacing each pair of linearized Pfaffians with a polynomial in traces using the relations described in Theorem 3.2[11], we get a polynomial in the traces which is an O_{2n}-invariant. Hence R_e is in the given FFG-ideal. Next, if R_o is an odd relation, then R_o^2 is an even relation, hence is in the given FFG-ideal. Then R_o is in the radical of the given FFG-ideal. $\square$

Then as in the odd dimension case, restricting to $k[SO_{2n}^\bullet]^{\mathrm{Ad}SO_{2n}}$, we find that $k[SO_{2n}^\bullet]^{\mathrm{Ad}SO_{2n}}$ is generated as an FFG-algebra by tr and pl, and the relations between these generators are generated, up to radical, by the relations for $k[O_{2n}^\bullet]^{\mathrm{Ad}O_{2n}}$, the relation described in [11, Theorem 3.2], and the relation det $= 1$ expressed in terms of tr.

Definition 3.16. An (even-dimensional) SO_{2n}-pseudocharacter of G over k is a pair of functions $T : G \to k$, $P : G^n \to k$, such that:

- T is an O_{2n}-pseudocharacter of G over k
- For all $g \in G$, $\det(T)(g) = 1$
- For all $g_1, \dots, g_n, h_1, \dots, h_n$, $P(g_1, \dots, g_n)P(h_1, \dots, h_n)$ satisfies the relation in [11, Theorem 3.2] with P in place of Q and T in place of tr.

Then we have the usual result.

4 Application: Conjugacy vs. Element-Conjugacy

In this section, we use our pseudocharacters to answer questions about conjugacy vs. element-conjugacy of group homomorphisms $\Gamma \to H$ for H a linear algebraic group, following Larsen [6, 7].

Definition 4.1. Fix a linear algebraic group H, and let Γ be another group. Two homomorphisms $\rho_1, \rho_2 : \Gamma \to H$ are called globally conjugate if there exists $h \in H$ such that $\rho_1 = h\rho_2 h^{-1}$. They are called element-conjugate if for all $\gamma \in \Gamma$, there exists $h \in H$ such that $\rho_1(\gamma) = h\rho_2(\gamma)h^{-1}$.

The "conjugacy vs. element-conjugacy" question for H asks whether or not element-conjugate semisimple homomorphisms $\Gamma \to H$ are automatically globally conjugate.

Definition 4.2. A group H is acceptable if element-conjugacy implies global conjugacy for all semisimple representations of arbitrary groups Γ. We call H finite-acceptable if element-conjugacy implies global conjugacy for all finite groups Γ, and compact-acceptable if element-conjugacy implies conjugacy for continuous semisimple representations of all compact groups Γ.

In [6, 7], Larsen mostly classifies the complex and compact simple Lie groups as finite-acceptable or finite-unacceptable (which implies unacceptable). Recent results show that $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, and their real compact forms are in fact compact-acceptable [3].

In this section, we give a simple sufficient condition for the acceptability of a connected reductive group H over $\mathbb{C}$, in terms of the FFG-algebra $k[H^\bullet]^{\mathrm{Ad}H}$, and we conjecture that this condition is also necessary (Conjecture 4.4). We then use this condition and the results of Section 3 to immediately show that $GO_n(\mathbb{C})$, $O_n(\mathbb{C})$, $GSp_{2n}(\mathbb{C})$, and $Sp_{2n}(\mathbb{C})$ are acceptable (not just finite- or compact-acceptable). By [6, Proposition 1.7], this implies that the compact forms of these groups are compact-acceptable, a result which was proviously known only for $O_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$.

We also use our pseudocharacters for $SO_{2n}(\mathbb{C})$ to give a criterion for when a semisimple representation $\rho : \Gamma \to SO_{2n}(k)$ is one half of a counterexample to acceptability for $SO_{2n}(k)$. We then construct a counterexample to acceptability for $SO_{2n}(\mathbb{C})$ ($n \geq 3$) for the domain group $\Gamma = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$; this gives a simpler example than the one in [6, Proposition 3.8], and it additionally shows that $SO_6(\mathbb{C})$ is unacceptable, a result which was not previously known.

4.1 General Principles

Let H be a linear algebraic group. Suppose that H has pseudocharacters consisting of one-argument functions only. More formally, let k be an algebraically closed field of characteristic 0, and suppose that there exist invariants $f_1, \ldots, f_n \in k[H]^{\mathrm{Ad}H}$ such that for any group Γ, the map $\rho \mapsto (f_1(\rho), \ldots, f_n(\rho))$ induces a bijection between

$$\{H(k)\text{-conjugacy classes of semisimple representations } \rho : \Gamma \to H(k)\}$$

and

$$\{\text{Maps } F_1, \ldots, F_n : \Gamma \to k \text{ satisfying certain fixed relations}\}.$$

Then $H(k)$ is acceptable: indeed, if $\rho_1, \rho_2 : \Gamma \to H(k)$ are semisimple element-conjugate representations, then for all $\gamma \in \Gamma$, we have $f_i(\rho_1(\gamma)) = f_i(\rho_2(\gamma))$ for $1 \leq i \leq n$, so ρ_1 and ρ_2 have the same H-pseudocharacter, hence they are conjugate.

When H is connected reductive (or, more generally, when the conclusion of Corollary 2.13 holds for H, such as for $H = O_n$), we can restate this result as follows.

Theorem 4.3. *Let H be an algebraic group over an algebraically closed field k of characteristic 0 such that Corollary 2.13 holds for H (e.g., H is connected reductive). Suppose that $k[H^\bullet]^{\mathrm{Ad}H}$ is generated by $k[H]^{\mathrm{Ad}H}$ as an* FFG*-algebra. Then $H(k)$ is acceptable.*

Of the connected reductive algebraic groups over $\mathbb{C}$ which are shown to be finite-acceptable in [6], all but $G_2(\mathbb{C})$ are known to satisfy the converse of this theorem, i.e., $\mathbb{C}[H^\bullet]^{\mathrm{Ad}H}$ is generated by $\mathbb{C}[H]^{\mathrm{Ad}H}$ as an FFG-algebra. We have not succeeded in determining whether or not $\mathbb{C}[G_2^\bullet]^{\mathrm{Ad}G_2}$ is generated by $\mathbb{C}[G_2]^{\mathrm{Ad}G_2}$, although computations with Mathematica show that $\mathbb{C}[M_7^\bullet]^{\mathrm{Ad}G_2}$ is not generated by $\mathbb{C}[M_7]^{\mathrm{Ad}G_2}$ (in constrast to the situation for GO_n, O_n, GSp_{2n}, and Sp_{2n}).

Nonetheless, we make the following conjecture:

Conjecture 4.4. *Let H be a connected reductive group over an algebraically closed field k of characteristic 0. Then $H(k)$ is acceptable iff $k[H^\bullet]^{\mathrm{Ad}H}$ is generated by $k[H]^{\mathrm{Ad}H}$ as an* FFG-*algebra.*

4.2 Element-conjugacy vs. Conjugacy for SO_{2n}

Let k be an algebraically closed field of characteristic 0, and let $n \geq 2$ be an integer. We wish to characterize all pairs of semisimple representations $\rho_1, \rho_2 : \Gamma \to SO_{2n}(k)$ which are element-conjugate but not globally conjugate. Let pl denote the linearized antisymmetrized Pfaffian (see Section 3.3 above). Our result is as follows.

Proposition 4.5. *Let Γ be a group, and let $\rho : \Gamma \to SO_{2n}(k)$ be a representation. Then there exists a representation $\rho' : \Gamma \to SO_{2n}(k)$ which is element-conjugate but not globally conjugate to ρ iff:*

- *For all* $\gamma \in \Gamma$, $\det(\rho(g) - \rho(g)^t) = 0$
- *There exist* $\gamma_1, \ldots, \gamma_n \in G$ *such that* $\mathrm{pl}(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \neq 0$.

In this situation, there is a unique such ρ' *up to conjugation by* $SO_{2n}(k)$*, and it is given by*

$$\rho'(\gamma) = X\rho(\gamma)X^{-1}$$

for any fixed $X \in O_{2n}(k) \setminus SO_{2n}(k)$.

Proof. **Uniqueness:** Let ρ' be element-conjugate but not globally conjugate to ρ in SO_{2n}. Then ρ and ρ' are element-conjugate in O_{2n}, hence globally conjugate in O_{2n}. Thus there is an $X \in O_{2n}(k)$ such that $\rho' = X\rho X^{-1}$. Since ρ and ρ' are not globally conjugate, conjugation by X must induce an outer automorphism of SO_{2n}. Since $SO_{2n}(k)$ has index 2 in $O_{2n}(k)$, easily any other choice of X gives a representation which is conjugate to ρ' in $SO_{2n}(k)$.

Existence, ($\Longrightarrow$): Let ρ' be a semisimple representation which is element-conjugate but not globally conjugate to $SO_{2n}(k)$. By the uniqueness proof, we can write $\rho' = X\rho X^{-1}$ as above. Let pl denote the linearized antisymmetrized Pfaffian, which is an odd n-ary invariant of $k[SO_{2n}^{\mathbb{N}}]^{\mathrm{Ad} SO_{2n}}$. Here "odd" means that

$$\mathrm{pl}(\rho(\gamma_1), \ldots, \rho(\gamma_n)) = -\mathrm{pl}(\gamma\rho(\gamma_1)X^{-1}, \ldots, X\rho(\gamma_n)X^{-1}) = -\mathrm{pl}(\rho'(\gamma_1), \ldots, \rho'(\gamma_n))$$

for all $\gamma_1, \ldots, \gamma_n \in \Gamma$. Since ρ and ρ' are not globally conjugate, they must have different pseudocharacters, and since $\mathrm{tr}(\rho) = \mathrm{tr}(\rho')$ by element-conjugacy, there must exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that

$$\mathrm{pl}(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \neq \mathrm{pl}(\rho'(\gamma_1), \ldots, \rho'(\gamma_n)).$$

Then by the above equation, $\mathrm{pl}(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \neq 0$.

Next, since ρ and ρ' are element-conjugate, $\rho|_{\langle\gamma\rangle}$ is conjugate to $\rho'|_{\langle\gamma\rangle}$ in SO_{2n} for each $\gamma \in \Gamma$, so

$$\mathrm{pl}(\rho(\gamma^{m_1}), \ldots, \rho(\gamma^{m_n})) = \mathrm{pl}(\rho'(\gamma^{m_1}), \ldots, \rho(\gamma^{m_n}))$$

for all $\gamma \in \Gamma$ and $m_1, \ldots, m_n \in \mathbb{Z}$. Hence by the above equation, $\mathrm{pl}(\rho(\gamma^{m_1}), \ldots, \rho(\gamma^{m_n})) = 0$. In particular, $\widetilde{\mathrm{pf}}(\rho(\gamma)) = \frac{1}{n!}\mathrm{pl}(\rho(\gamma), \ldots, \rho(\gamma)) = 0$ for all $\gamma \in \Gamma$. Hence

$$\det(\rho(\gamma) - \rho(\gamma)^t) = \mathrm{pf}(\rho(\gamma) - \rho(\gamma)^t)^2 = \widetilde{\mathrm{pf}}(\rho(\gamma))^2 = 0.$$

Existence, ($\Longleftarrow$): Let $X \in O_{2n}(k) \setminus SO_{2n}(k)$, and set $\rho'(\gamma) = X\rho(\gamma)X^{-1}$. Then by assumption, there exist $\gamma_1, \ldots, \gamma_n$ such that

$$\mathrm{pl}(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \neq -\mathrm{pl}(\rho(\gamma_1), \ldots, \rho(\gamma_n)) = \mathrm{pl}(\rho'(\gamma_1), \ldots, \rho'(\gamma_n)),$$

so ρ and ρ' are not globally conjugate.

Now fix $\gamma \in \Gamma$. To show that $\rho|_{\langle\gamma\rangle}$ and $\rho'_{\langle\gamma\rangle}$ are conjugate in SO_{2n}, it suffices to show that they have the same SO_{2n}-pseudocharacters. They have the same traces because ρ and ρ' are conjugate in O_{2n}. To show that they have the same linearized Pfaffians, we must show

$$\mathrm{pl}(\rho(\gamma^{m_1}), \ldots, \rho(\gamma^{m_n})) = 0$$

for all $m_1, \ldots, m_n \in \mathbb{Z}$, since the corresponding Pfaffian for ρ' is the negative of that for ρ. By definition, $\mathrm{pl}(\rho(\gamma^{m_1}), \ldots, \rho(\gamma^{m_n}))$ is the multilinear term in

$$\widetilde{\mathrm{pf}}(t_1\rho(\gamma^{m_1}) + \cdots + t_n\rho(\gamma^{m_n})) = \mathrm{pf}(t_1(\rho(\gamma^{m_1}) - \rho(\gamma^{m_1})^t) + \cdots + t_n(\rho(\gamma^{m_n}) - \rho(\gamma^{m_n})^t).$$

But $\rho(\gamma) - \rho(\gamma)^t = \rho(\gamma) - \rho(\gamma)^{-1}$ divides $\rho(\gamma)^{m_i} - \rho(\gamma)^{-m_i} = \rho(\gamma^{m_i}) - \rho(\gamma^{m_i})^t$ for all i, so the assumption $\det(\rho(\gamma) - \rho(\gamma)^t) = 0$ implies that

$$\det(t_1(\rho(\gamma^{m_1}) - \rho(\gamma^{m_1})^t) + \cdots + t_n(\rho(\gamma^{m_n}) - \rho(\gamma^{m_n})^t) = 0.$$

Hence taking the square root, the Pfaffian is zero as well for all values of $t_1, \dots, t_n$. Thus $\mathrm{pl}(\rho(\gamma^{m_1}), \dots, \rho(\gamma^{m_n})) = 0$, proving the claim. □

4.3 A Finite Abelian Counterexample for SO_{2n}, $n \geq 3$

Let $\Gamma = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, with generators $(1, 0)$ and $(0, 1)$. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO_2(\mathbb{C})$$

Define a homomorphism $\rho_6 : \Gamma \to SO_6(\mathbb{C})$ by:

$$\begin{aligned} \rho_6(1, 0) &= A \oplus A \oplus I \\ \rho_6(0, 1) &= I \oplus A \oplus A \end{aligned}$$

Then one can check that $\det(\rho_6(\gamma) - \rho_6(\gamma)^t) = 0$ for all $\gamma \in \Gamma$, while $\mathrm{pl}(\rho_6(1, 0), \rho_6(0, 1), \rho_6(0, 1)) = 16$. Hence ρ_6 is a counterexample to element-conjugacy implying conjugacy for $SO_6(\mathbb{C})$.

More generally, we have:

Proposition 4.6. *Let Γ and A be as above. For any $n \geq 3$, the homomorphism $\rho_{2n} : \Gamma \to SO_{2n}(\mathbb{C})$ defined by*

$$\begin{aligned} \rho_{2n}(1, 0) &= A \oplus A \oplus I \oplus \bigoplus_{i=4}^{n} A \\ \rho_{2n}(0, 1) &= I \oplus A \oplus A \oplus \bigoplus_{i=4}^{n} A \end{aligned}$$

satisfies $\det(\rho_{2n}(\gamma) - \rho_{2n}(\gamma)^t) = 0$ *for all* $\gamma \in \Gamma$ *and* $\mathrm{pl}(\rho_{2n}(1, 0), \rho_{2n}(0, 1), \dots, \rho_{2n}(0, 1)) \neq 0$. *Hence ρ_{2n} gives a counterexample to element-conjugacy implying global conjugacy.*

Proof. We have

$$\begin{aligned} \det\left(\left(\bigoplus_{i=1}^{n} B^{(i)}\right) - \left(\bigoplus_{i=1}^{n} B^{(i)}\right)^t\right) &= \det\left(\bigoplus_{i=1}^{n} (B^{(i)} - (B^{(i)})^t)\right) \\ &= \prod_{i=1}^{n} \det(B^{(i)} - (B^{(i)})^t) \end{aligned}$$

Hence to show $\det(\rho(\gamma) - \rho(\gamma)^t) = 0$, it suffices to prove that some 2×2 diagonal block $B^{(i)}$ of $\rho(\gamma)$ satisfies $\det(B^{(i)} - (B^{(i)})^t) = 0$. But one can check that for all $\gamma \in \Gamma$, one of the first three 2×2 diagonal blocks is a symmetric matrix.

Next, recall that $\mathrm{pl}(B_1, \ldots, B_n)$ is defined to be the coefficient of $t_1 \cdots t_n$ in $\mathrm{pf}(t_1(B_1 - B_1^t) + \cdots + t_n(B_n - B_n^t))$. Letting each $B_j = \bigoplus_{i=1}^n B_j^{(i)}$ for some 2×2 matrices $B_j^{(i)}$, we have

$$\mathrm{pf}(t_1(B_1 - B_1^t) + \cdots + t_n(B_n - B_n^t)) = \prod_{i=1}^n \mathrm{pf}(t_1(B_1^{(i)} - (B_1^{(i)})^t) + \cdots + t_1(B_n^{(i)} - (B_n^{(i)})^t)$$

Now pf is a linear function of 2×2 antisymmetric matrices, so this equals

$$\prod_{i=1}^n \sum_{j=1}^n t_j \mathrm{pf}(B_j^{(i)} - (B_j^{(i)})^t)$$

Taking the coefficient of $t_1 \ldots t_n$ in this formula, we find that

$$\mathrm{pl}(B_1, \ldots, B_n) = \sum_{\sigma \in S_n} \prod_{i=1}^n \mathrm{pf}(B_{\sigma(i)}^{(i)} - (B_{\sigma(i)}^{(i)})^t)$$

Finally, note that $\mathrm{pf}(A - A^t) = 2$ and $\mathrm{pf}(I - I^t) = 0$. Thus

$$\mathrm{pl}(C_1 := \rho_{2n}(1,0), C_2 := \rho_{2n}(0,1), \ldots, C_n := \rho_{2n}(0,1))$$

will be nonzero so long as for some $\sigma \in S_n$, for all i, $C_{\sigma(i)}^{(i)} = A$. Taking σ to be the identity permutation works. $\square$

Corollary 4.7. *For all $n \geq 3$ and all odd primes p such that $\left(\frac{-1}{p}\right) = 1$, there is a continuous semisimple representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to SO_{2n}(\mathbb{C})$ which is a counterexample to acceptability.*

Proof. In light of the above proposition and Maschke's theorem, it suffices to prove that there is an extension K of $\mathbb{Q}_p$ with $\mathrm{Gal}(K/\mathbb{Q}_p) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. By assumption, $\mu_4 \subset \mathbb{Q}_p$, so Kummer theory tells us that this will be true iff $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^4$ has a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. By the theory of local fields (see, e.g., Proposition 6.8[8]), we have

$$\left|\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^4\right| = 16$$
$$\left|\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2\right| = 4.$$

Hence $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^4 \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, so adjoining all fourth roots to $\mathbb{Q}_p$ gives the desired extension K. $\square$

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